

# Lecture notes .in

## CHAPTER # 1 INTRODUCTION TO DIGITAL CONTROL SYSTEMS

After completing this chapter, the students will be able to:

- Model the digital system,
- Find the Z transform, basic theories related to Z-transform and methods of finding inverse Z transform,
- Solve difference equation by Z transform.

### 1. Introduction

Computers can be used to control the position of the antenna (Plant) given in Fig. 1 below. Such system is called computer-controlled system or sampled-data system or discrete system or digital system.

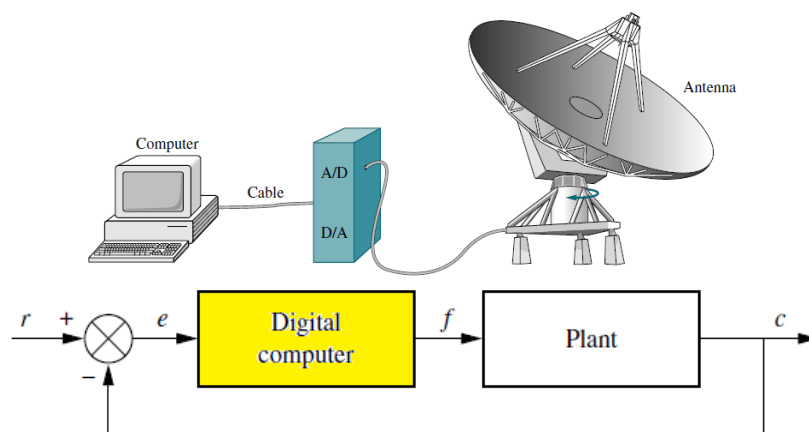


Fig. 1 Computer-controlled system

The signals  $r$ ,  $e$  and  $c$  are analog but the signal  $f$  is digital. For signal matching, analog to digital (A/D) converter and digital to analog (D/A) converter must be used as shown in Fig. 2.

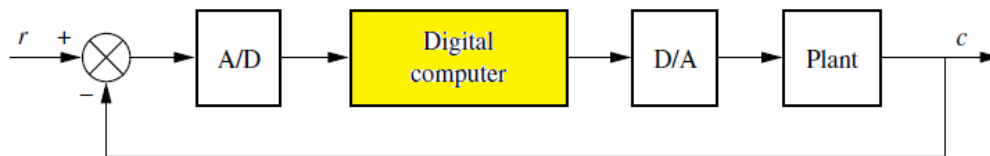


Fig. 2 Placement of A/D and D/A converters

## 2. Analog to Digital (A/D) Converter

A/D converter is a two-step process and is not instantaneous. There is a delay between the input analog voltage and the output digital word.

**First Step:** the analog signal is converted to a sampled signal at periodic intervals to  $2^n$  levels ( $n \neq 0$ ,  $n=1$  or  $2 \dots$  etc);

**Second Step:** Based on the sampling rate, the sampled signal is converted to a sequence of binary numbers, that represent the digital signal.

The sampling rate must be at least twice the bandwidth of the signal, or there will be distortion. The minimum sampling frequency is called the Nyquist sampling rate.

To explain these two steps, consider the analog signal shown in Fig. 3 (a). This analog signal is sampled to 8 quantization levels (3-bit). Each level is assigned with 3-digit number started with 000 and end with 111 as shown in Fig. 3 (b). (Note that the analog signal sampled at periodic intervals and held over the sampling interval by a device called a zero-order sample-and-hold (Z.O.H.) that yields a staircase approximation to the analog signal. (Higher order hold like first order hold generates a ramp between the samples). From Fig. 3 (b), we can see that there will be an associated error for each digitized analog value. This is called the quantization error which occurs due to the quantization

process rounds off the analog voltage to the next higher or lower level.

After sampling and holding, the A/D converter converts the sample to a digital number (as shown in Fig 3 (c)).

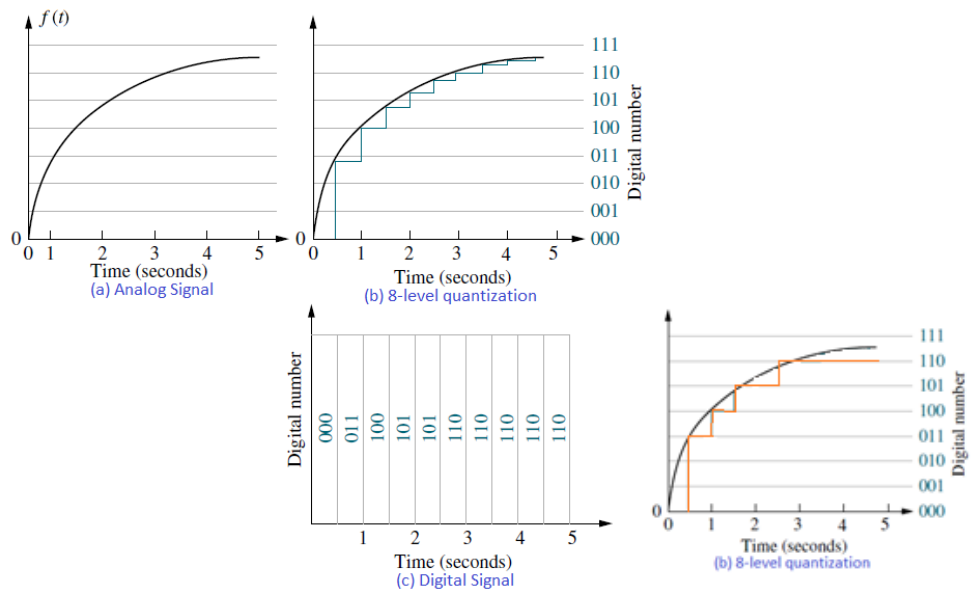
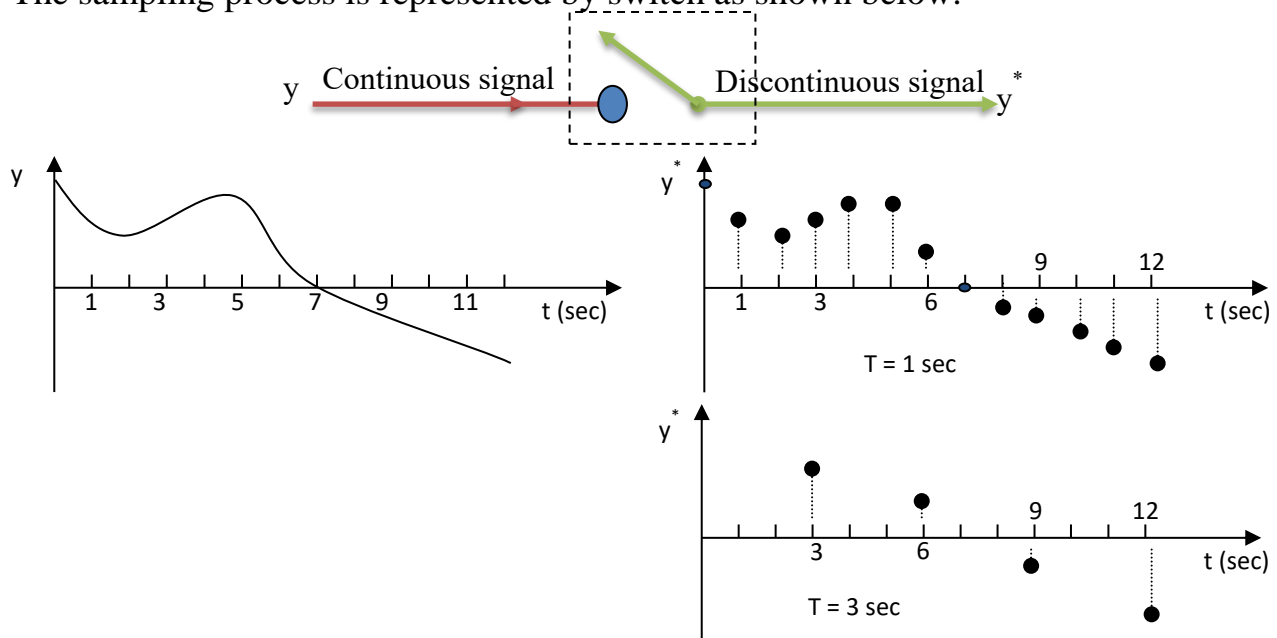


Fig. 3 A/D converter process (sampling time is 0.5 second)

Note that, the leftmost bit of any binary code has the most weight (one half of the Full Scale [FS]) and is called the Most Significant Bit (MSB). On the other hand, the rightmost bit has the least weight and is called the Least Significant Bit (LSB). The quantization level ( $Q$ ) is always equal the LSB. Moreover, the quantization error varies between 0 and  $\pm Q$ . To reduce that error, we need to increase the number of quantization levels (this is achieved by increasing the number of bits).

The sampling process is represented by switch as shown below.



$$y^*(t) = y(0)\delta(t) + y(T)\delta(t - T) + y(2T)\delta(t - 2T) + \dots + y(kT)\delta(t - kT)$$

$$y^*(t) = \sum_{k=0}^{\infty} y(kT)\delta(t - kT)$$

1. At sampling times, strength of impulse is equal to value of input signal.
2. Between sampling times, it is zero.

If we take Laplace transform of both sides of the previous equation, then:

$$Y^*(s) = y(0) \mathcal{L}\{\delta(t)\} + y(T)\mathcal{L}\{\delta(t - T)\} + y(2T) \mathcal{L}\{\delta(t - 2T)\} + \dots$$

$$Y^*(s) = y(0) + y(T)e^{-Ts} + y(2T) e^{-2Ts} + \dots$$

$$Y^*(s) = \sum_{k=0}^{\infty} y(kT)e^{-kTs}$$

If we define  $z = e^{Ts}$

$$Y^*(s) = \sum_{k=0}^{\infty} y(kT)z^{-k}$$

This will be explained later as z transforms.

### 3. Digital to Analog (D/A) Converter:

D/A converter process is simple and instantaneous. The weighted voltages are summed together to yield the analog output. For example, in Fig. 4, three weighted voltages are summed. The three-bit binary code is represented by electronic switches. Thus, if the binary number is 110, the center and bottom switches are on, and the analog output is 6 volts.

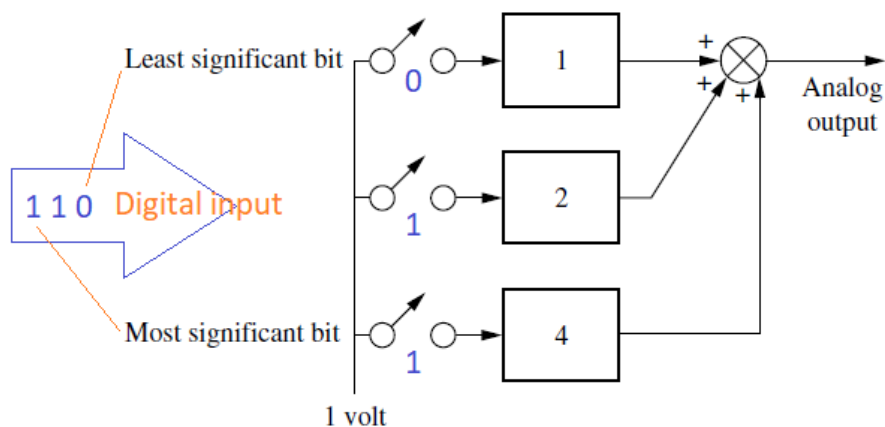


Fig. 4 D/A process

In order to eliminate the zero value between sampling interval, a Holder device is used.

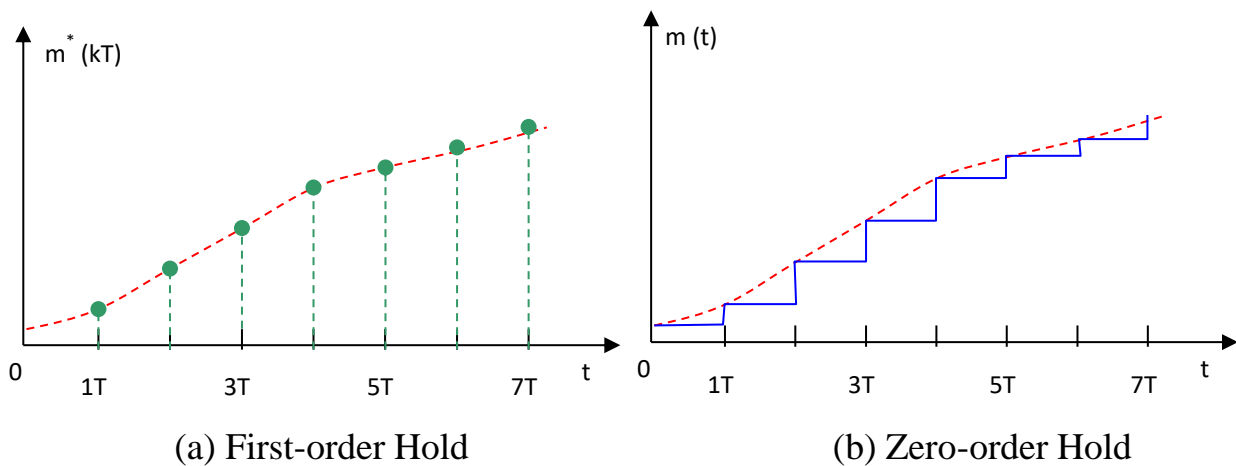


Fig. 5 Types of holder elements

According to the slope of the line connecting between two successive points, the holder took its name. For example, the slope of line in Fig. 5(b) is zero; therefore, the hold element is called zero-order holder. Moreover, the slope of line in Fig. 5(a) is linear; therefore, the hold element is called first-order holder.

#### 4. Modelling of Sampler

The objective at this point is to derive a mathematical model for the digital computer as represented by a sampler and zero-order hold. Consider the sampling model shown in Fig. 6, where the sampler  $S(t)$  is represented by two step functions (one +ve and another -ve) as shown in figure 6:

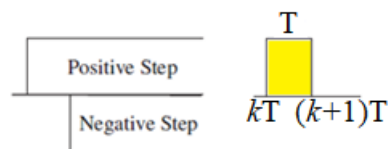


Fig. 6 Pulse train representation of sampler

$$S(t) = \sum_{k=0}^{\infty} u(t - kT) - u(t - (k + 1)T)$$

Where  $k$  is an integer  $0, 1, 2, \dots$  etc

$T$  is the period of the pulse train that represent the sampling period.

$$f^*(t) = f(t) \times \sum_{k=0}^{\infty} u(t - kT) - u(t - (k + 1)T)$$

Since the sampled signal  $f^*(t)$  is only defined at  $t = kT$ , then we can rewrite the above equation as:

$$f^*(t) = \sum_{k=0}^{\infty} f(kT) \{u(t - kT) - u(t - (k + 1)T)\}$$

Taking Laplace

$$F^*(s) = \sum_{k=0}^{\infty} f(kT) \left\{ \frac{e^{-kTs} - e^{-(k+1)Ts}}{s} \right\}$$

$$F^*(s) = \sum_{k=0}^{\infty} f(kT) e^{-kTs} \left\{ \frac{1 - e^{-Ts}}{s} \right\} \quad (1)$$

The above equation has two terms:

The first term  $\sum_{k=0}^{\infty} f(kT) e^{-kTs}$  represents the samples function. But the second term  $\frac{1 - e^{-Ts}}{s}$  represents the zero-order hold as shown in Fig. 7.

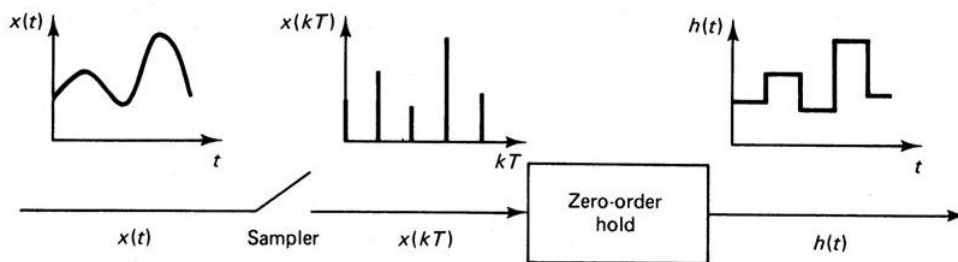


Fig. 7 Sampler with zero-order hold

Based on the Laplace transform of the sampled signal given by eqn. (1), then the zero-order hold is represented by:

$$Z.O.H = \frac{1 - e^{-Ts}}{s}$$

In this chapter, we will assume we are sampling the continuous time signal  $x(t)$  at a uniform sampling rate. The time interval between samples will be denoted by  $T$ . Thus, we will denote the discrete-time (sampled) values as  $x(0), x(T), x(2T); \dots; x(kT)$ . This is shown graphically in Fig. 8. Sometimes we remove the explicit dependence on the

sampling interval  $T$  and just write  $x(0), x(1), x(2), \dots, x(k)$ , since the sample interval is the same for all of the discrete samples.

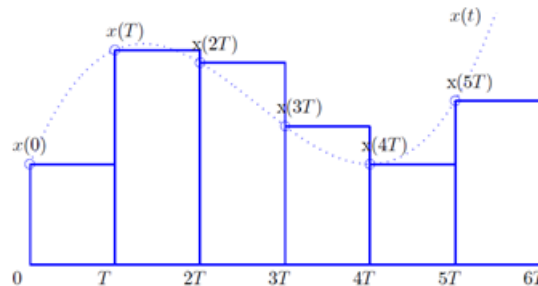
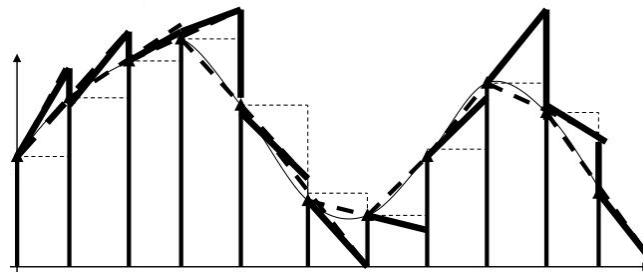


Fig. 8 Analog signal  $x(t)$  and Discrete signal  $x(kT)$

On the other hand, if first-order holder is used, its transfer function will be:

$$\left( \frac{1 - e^{-Ts}}{s} \right)^2 \frac{Ts + 1}{T}$$



## 5. Difference Equation and Differential Equation

### A differential equation

- Expresses the change in some variable as a result of an *infinitesimal* change in another variable.
- Dealing with variables which are **functions of time**.

### A difference equation

- Expresses the change in some variable as a result of a *finite* change in another variable.
- **Time discretization**, (sequences of discrete numbers).
- These discrete numbers may come from **sampling a continuous variable**, or they may be **generated within a computer**.

Consider the mass-spring-damper shown in Fig. 8, the differential equation that represents the displacement  $y(t)$  is:

$$F(t) = m\ddot{y}(t) + B\dot{y}(t) + Ky(t)$$

Where  $F$  is applied force,  $m$  is the mass,  $B$  is damping constant, and  $K$  is spring constant.

Rearranging above equation in following form

$$\ddot{y}(t) = \frac{1}{m}F(t) - \frac{B}{m}\dot{y}(t) - \frac{K}{m}y(t)$$

The block diagram representing the above differential equation is shown in Fig. 9.

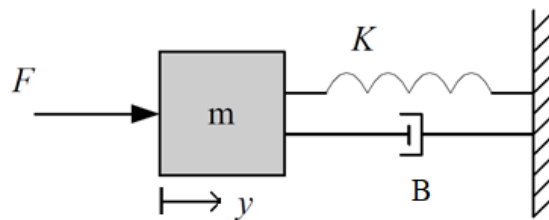


Fig. 8 Mass-spring-damper system

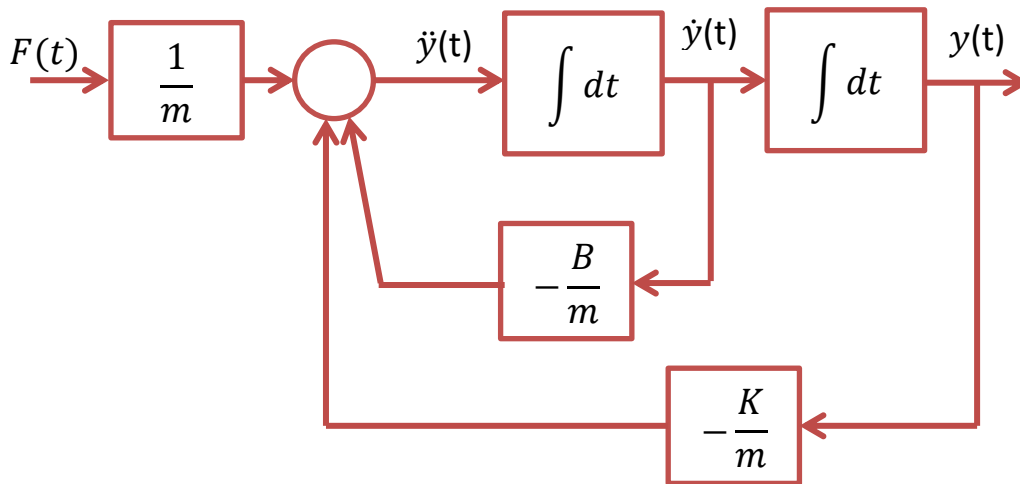


Fig. 9 Block diagram representing the differential equation

The differential equation mentioned above is converted to difference equation as:

$$y(k+2) = \frac{1}{m}F(k) - \frac{B}{m}y(k+1) - \frac{K}{m}y(k)$$

The block diagram representing the above difference equation is shown in Fig. 10.



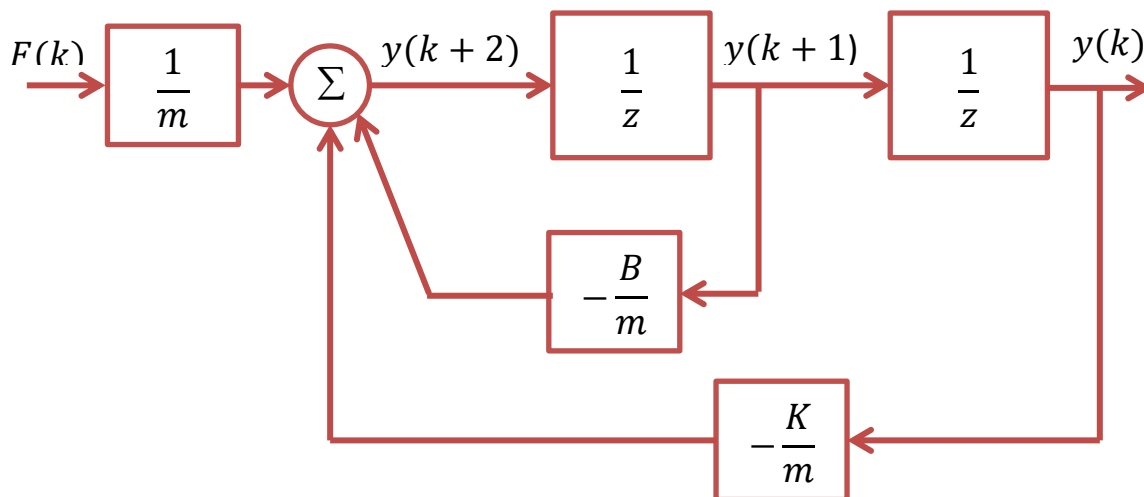
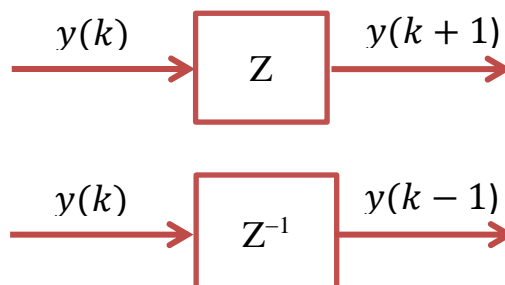


Fig. 10 Block diagram representing the difference equation

Where the term  $Z^{-1}$  represents the delay acting on sequence.



**Example:**

Consider the difference equation given below

$$y(k + 3) + 2y(k + 2) + 4y(k + 1) - y(k) = u(k + 1) + 3u(k)$$

What is the order of this difference equation? Is this equation linear or nonlinear, time variant or time invariant, homogenous or non-homogenous?

The above equation is 3<sup>rd</sup> order due to the (k+3) term,

The above equation is Linear and Time invariant, because all coefficients are constants (1, 2, 4 and -1),

The above equation is non-homogenous, as the coefficients regarding the input (u) are nonzero



### Example

For each of the following difference equations, determine the order of the equation. Is the equation (a) linear, (b) time invariant, or (c) homogeneous?

1.  $y(k + 2) + 0.8y(k + 1) + 0.07y(k)u(k)$
2.  $y(k + 4) + \sin(0.4k)y(k + 1) + 0.3y(k) = 0$
3.  $y(k + 1) = -0.1y^2(k)$

### Solution

1. The equation is second order. All terms enter the equation linearly and have constant coefficients. The equation is therefore LTI. A forcing function appears in the equation, so it is nonhomogeneous.
2. The equation is fourth order. The second coefficient is time dependent, but all the terms are linear and there is no forcing function. The equation is therefore linear time varying and homogeneous.
3. The equation is first order. The right-hand side (RHS) is a nonlinear function of  $y(k)$  but does not include a forcing function or terms that depend on time explicitly. The equation is therefore nonlinear, time invariant, and homogeneous.

In the next section, we start to create a transform approach to digital systems by introducing the z-transform.

## 6. Z-Transform

For discrete signals Laplace transform can't be used. Laplace transform is used with analog signals. For discrete signals, Laplace transform is replaced by another related transform called the z-transform. The z-transform will be used for discrete signals.

As Laplace transform converts the differential equations into algebraic terms with respect to  $s$ , z-transform converts the difference equations into algebraic terms with respect to  $z$ . Both transformations are matching a complex quantity to the points of a region of the complex plane.

It should be noted that the z-plane (i.e., the domain of z-transform) is organized in a polar form, while the s-plane (i.e., the domain of Laplace transform) is in a Cartesian form as shown in Fig. 11.

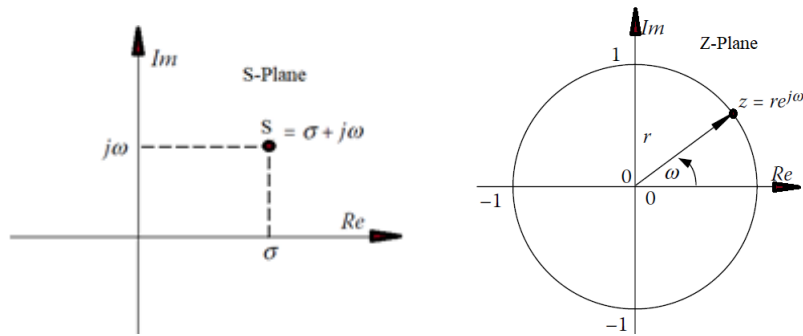


Fig. 11, Cartesian plane of S and polar plane of Z

$$X(s) = \int_{t=0}^{\infty} x(t) \cdot e^{st} \cdot dt$$

By replacing the time (t) in continuous domain by (kT) in discrete domain and considering unity sampling (T=1), we get:

$$X(z) = \sum_{k=0}^{k=\infty} x[kT] \cdot Z^{-kT} = x_0 + x_1 Z^{-1} + x_2 Z^{-2} + \dots$$

The above equation is a valuable tool for analyzing discrete linear time-invariant (LTI) systems and it is called z-transform.

The relation between S-plane and Z-plane is governed by:

$$Z = e^{sT}$$

Replacing S by  $\sigma + j\omega$  and assuming the sampling period (T) is 1

$$Z = r e^{j\omega} = e^{\sigma + j\omega} = e^{\sigma} \cdot e^{j\omega}$$

$$r = e^{\sigma} \text{ or } \sigma = \ln(r)$$

As a consequence of the above differences, vertical lines in s-plane become circles in z-plane. Therefore, the imaginary axis of s-plane, that is, the line formed by setting  $\sigma = 0$ , will reflect to a circle of radius  $r = 1$  in the z-plane. So, when  $\sigma = 0$  in s-plane, corresponds to  $r = 1$  in z-plane. For any point located in the left half of s-plane ( $\sigma < 0$ ) corresponds to a point located within the unit circle in the z-plane. A similar behavior is realized for any point located in the right half of s-plane will give a point located outside the unit circle in the z-plane as explained in Fig. 12.

For example, a continuous-time system is *stable* if its poles are in the left s-half-plane. Correspondingly, a discrete-time system is stable if its poles are located inside the unit circle of z-plane as it will explain in chapter 5.

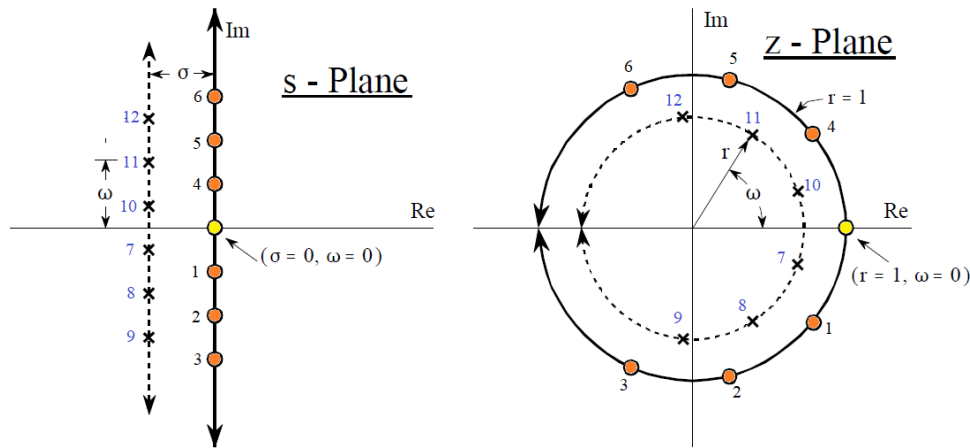
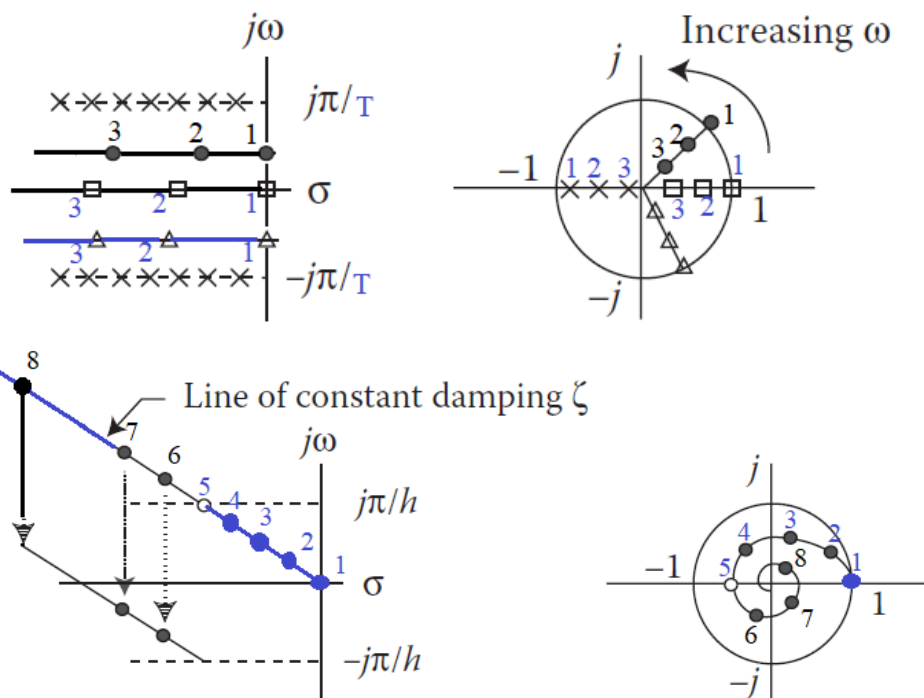


Fig. 12 Mapping from S-plane to Z-plane

We must pay attention to how the frequency variable,  $\omega$ , is used in the two transforms. In continuous S plane, the frequency varies from 0 to  $+\infty$  in upper half, and from 0 to  $-\infty$  in the lower half. In comparison, a discrete Z plane can only have a frequency between 0 and  $\pi$  in the upper half and between 0 and  $-\pi$  in the lower half. That is, the frequency must be between 0 and 0.5 when expressed as a fraction of the sampling rate. This matches the geometry of the z-plane when we interpret  $\omega$  to be an angle expressed in radians.



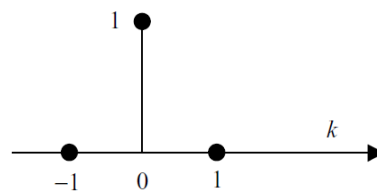


Finally, it is important to know that, z-transform is restricted to causal signals (i.e., signals with zero values for negative time) and the one-sided z-transform.

## 6.1 Z-transform of Standard Functions

### 6.1.1 Unit Impulse

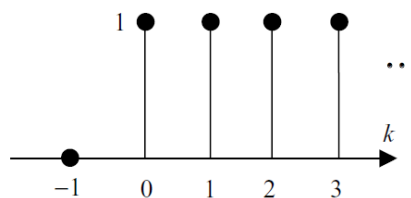
$$\delta(k) = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$$



$$\delta(z) = 1$$

### 6.1.2 Unit Step

$$u(k) = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases}$$



$$U(z) = 1 + z^{-1} + z^{-2} + z^{-3} + \dots + z^{-k} = \sum_{k=0}^{\infty} z^{-k}$$

This is called the open-form solution

Recall that sum of an infinite geometric series is given by the formula

$$Sum = \frac{1st\ term}{1 - ratio\ between\ any\ two\ successive\ terms\ (r)}, |r| < 1$$

$$U(z) = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}, \quad |z| > 1$$

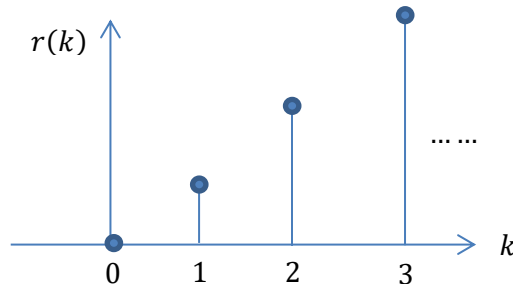
This is called the closed-form solution

For the above expression of U(z) (open form), at  $|z| \leq 1$ ,  $U(z) \rightarrow \infty$  (not converge). On the other hand, at  $|z| > 1$ ,  $U(z) \rightarrow 1$  (converge). Therefore, it is necessary to specify the region of (z) over which the function U(z) is convergent.



### 6.1.3 Unit Ramp

$$r(k) = \begin{cases} k, & k \geq 0 \\ 0, & k < 0 \end{cases}$$



$$r(k) = k \times 1$$

$$z\{k^m f(k)\} = \left(-z \frac{d}{dz}\right)^m F(z), \quad m = 1$$

$$z\{k^1 f(k)\} = \left(-z \frac{d}{dz}\right)^1 F(z)$$

$$f(k) = 1, \text{ then } f(z) = \frac{Z}{Z-1}$$

$$\mathcal{Z}\{k \times 1\} = \left(-z \frac{d}{dz}\right) \left(\frac{z}{z-1}\right) = (-z) \frac{(z-1) - z}{(z-1)^2} = \frac{z}{(z-1)^2}$$

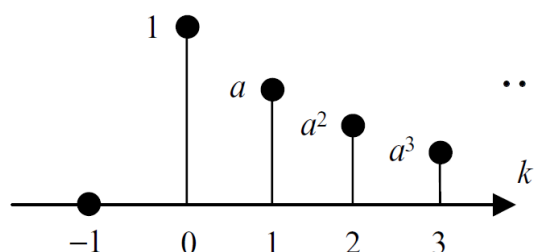
$$R(z) = \frac{z}{(z-1)^2}$$

In general, for a sampling time (T) rather than unity,

$$R(z) = \frac{Tz}{(z-1)^2}$$

### 6.1.4 Polynomial function

$$u(k) = \begin{cases} a^k, & k \geq 0 \\ 0, & k < 0 \end{cases}$$





$$U(z) = 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + \dots + a^kz^{-k} = \sum_{k=0}^{\infty} (a^{-1}z)^{-k}$$

$$U(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

### 6.1.5 Exponential function

$$u(k) = \begin{cases} e^{-akT}, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

$$U(z) = 1 + e^{-aT}z^{-1} + e^{-2aT}z^{-2} + e^{-3aT}z^{-3} + \dots + e^{-akT}z^{-k}$$

$$U(z) = \frac{1}{1 - e^{-aT}z^{-1}} = \frac{z}{z - e^{-aT}}$$

### 6.1.6 Sinusoidal/Cosinusoidal function

As we know

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

$$e^{-j\omega t} = \cos(\omega t) - j\sin(\omega t)$$

From these two equations;

$$\cos(\omega t) = \frac{1}{2}(e^{j\omega t} + e^{-j\omega t})$$

$$\sin(\omega t) = \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})$$

$$Z(\cos(\omega t)) = Z\left(\frac{1}{2}(e^{j\omega t} + e^{-j\omega t})\right) = Z\left(\frac{e^{j\omega t}}{2}\right) + Z\left(\frac{e^{-j\omega t}}{2}\right)$$

$$X(Z) = Z(\cos(\omega t)) = \frac{1}{2}\left\{\left(\frac{Z}{Z - e^{j\omega T}}\right) + \left(\frac{Z}{Z - e^{-j\omega T}}\right)\right\}$$

$$X(Z) = \frac{1}{2} \frac{Z(2Z - (e^{j\omega T} + e^{-j\omega T}))}{Z^2 - (e^{j\omega T} + e^{-j\omega T})Z + 1}$$

$$X(Z) = \frac{Z^2 - Z \frac{e^{j\omega T} + e^{-j\omega T}}{2}}{Z^2 - (e^{j\omega T} + e^{-j\omega T})Z + 1}$$

$$X(Z) = \frac{Z^2 - Z\cos(\omega T)}{Z^2 - 2Z\cos(\omega T) + 1}$$

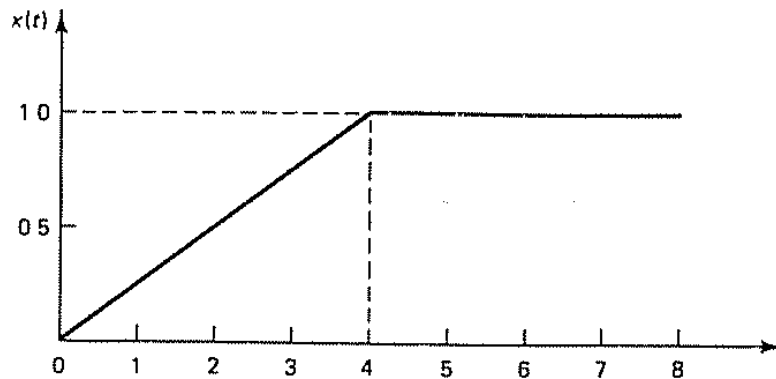


Similarly, we can obtain the Z-transform of sinusoidal function as:

$$X(Z) = Z(\sin(\omega t)) = \frac{Z \sin(\omega T)}{Z^2 - 2Z \cos(\omega T) + 1}$$

**Example:**

Obtain the z transform of the curve x(t) shown below assuming T=1 sec.



$$f_1(t) = 0.25 r(t) \times \{u(t) - u(t - 4)\}$$

$$f_1(t) = 0.25 r(t) \times u(t) - 0.25 r(t) \times u(t - 4)$$

$$f_1(t) = 0.25 r(t) \times u(t) - 0.25 r(t - 4 + 4) \times u(t - 4)$$

$$f_1(t) = 0.25 r(t) \times u(t) - 0.25 r(t - 4) \times u(t - 4) - u(t - 4)$$

$$f_1(t) = 0.25 r(t) - 0.25 r(t - 4) - u(t - 4)$$

$$f_2(t) = u(t - 4)$$

$$x(t) = f_1(t) + f_2(t) = 0.25 r(t) - 0.25 r(t - 4)$$

As we know, the z transform of ramp function is

$$\frac{Tz}{(z - 1)^2} = \frac{z}{(z - 1)^2} \quad (\text{as } T = 1)$$

Also, the z-transform of delayed ramp function is

$$z^{-4} \frac{Tz}{(z - 1)^2} = \frac{Tz^{-3}}{(z - 1)^2} = \frac{z^{-3}}{(z - 1)^2} \quad (\text{as } T = 1)$$

$$X(z) = \frac{0.25 z}{(z - 1)^2} + \frac{0.25 z^{-3}}{(z - 1)^2} = \frac{0.25z(1 + z^{-4})}{(z - 1)^2}$$





## 6.2 Properties of Z-transform

The properties of the  $z$ -transform have a very similar meaning to the properties of the Laplace transform. The most important properties of the  $z$ -transform, commonly used for problem solving, are presented below.

No.	Property	Formula
1	Linearity	$\mathcal{Z}\{\alpha f_1(k) + \beta f_2(k)\} = \alpha F_1(z) + \beta F_2(z)$
2	Time Delay	$\mathcal{Z}\{f(k-n)\} = z^{-n}F(z)$
3	Time Advance	$\mathcal{Z}\{f(k+1)\} = zF(z) - zf(0)$ $\mathcal{Z}\{f(k+n)\} = z^n F(z) - z^n f(0) - z^{n-1}f(1) \dots - zf(n-1)$
4	Discrete-Time Convolution	$\mathcal{Z}\{f_1(k)*f_2(k)\} = \mathcal{Z}\left\{\sum_{i=0}^k f_1(i)f_2(k-i)\right\} = F_1(z)F_2(z)$
5	Multiplication by Exponential	$\mathcal{Z}\{a^{-k}f(k)\} = F(az)$
6	Complex Differentiation	$\mathcal{Z}\{k^m f(k)\} = \left(-z \frac{d}{dz}\right)^m F(z)$
7	Final Value Theorem	$f(\infty) = \lim_{k \rightarrow \infty} f(k) = \lim_{z \rightarrow 1} (1-z^{-1})F(z) = \lim_{z \rightarrow 1} (z-1)F(z)$
8	Initial Value Theorem	$f(0) = \lim_{k \rightarrow 0} f(k) = \lim_{z \rightarrow \infty} F(z)$

### 6.2.1 Linearity

Suppose we have two functions along with their respective Z transforms:

$$x_1[k] \leftrightarrow X_1(z) \quad \text{and} \quad x_2[k] \leftrightarrow X_2(z)$$

The property of linearity means that

$$\mathcal{Z}(a x_1[k] + b x_2[k]) = a X_1(z) + b X_2(z)$$

Where  $a$  and  $b$  are constants.

#### Example:

$$f(k) = 2 \times 1(k) + 4 \times \delta(k), \quad k = 0, 1, 2, \dots$$

$$F(z) = \mathcal{Z}\{2 \times 1(k) + 4 \times \delta(k)\}$$

$$F(z) = 2 \times \mathcal{Z}\{1(k)\} + 4 \times \mathcal{Z}\{\delta(k)\}$$

$$F(z) = 2 \times \frac{z}{z-1} + 4 = \frac{6z-4}{z-1}$$



### 6.2.2 Time Reverse

Suppose we have a function with its Z transform:

$$x[k] \leftrightarrow X(z)$$

Then

$$x[-k] \leftrightarrow X\left(\frac{1}{z}\right)$$

### 6.2.3 Time Delay (Right Shift)

Suppose we have a function with its Z transform:

$$x[k] \leftrightarrow X(z)$$

Then

$$x[k - n] \leftrightarrow Z^{-n}X(z)$$

#### Example:

$$f(k) = \begin{cases} 4, & k = 2, 3, 4, \dots \\ 0, & \text{otherwise} \end{cases}$$

The given sequence is a sampled step starting at  $k=2$  rather than  $k=0$  (i.e. it is delayed by two sampling periods).

$$F(z) = \mathcal{Z}\{4 \times 1(k - 2)\} = 4 z^{-2} \mathcal{Z}\{1(k)\} = z^{-2} \frac{4z}{z - 1} = \frac{4}{z(z - 1)}$$

### 6.2.4 Time Advance (Left Shift)

$$x[k] \leftrightarrow X(z)$$

Then

$$x[k + 1] \leftrightarrow ZX(z) - ZX(0)$$

$$x[k + 2] \leftrightarrow Z^2X(z) - Z^2X(0) - ZX(1)$$

$$x[k + 3] \leftrightarrow Z^3X(z) - Z^3X(0) - Z^2X(1) - ZX(2)$$

$$x[k + 4] \leftrightarrow Z^4X(z) - Z^4X(0) - Z^3X(1) - Z^2X(2) - ZX(3)$$

In general

$$x[k + n] \leftrightarrow Z^nX(z) - Z^nX(0) - Z^{n-1}X(1) - Z^{n-2}X(2) - \dots - ZX(n - 1)$$

#### Example:

Using the time advance property, find the  $z$ -transform of the causal sequence

$$f(k) = \{4, 8, 16, 32, \dots\}$$

The sequence can be written as

$$f(k) = 2^{k+2} = y(k + 2) \quad \text{where } y(k) = 2^k$$

$$Y(z) = \frac{Z}{Z - 2}$$



From  $y(k) \rightarrow y(0) = 1$  and  $y(1) = 2$

$$y[k + 2] \leftrightarrow Z^2 Y(z) - Z^2 y(0) - Z y(1) = Z^2 \frac{Z}{Z - 2} - Z^2 - 2Z = \frac{4Z}{Z - 2}$$

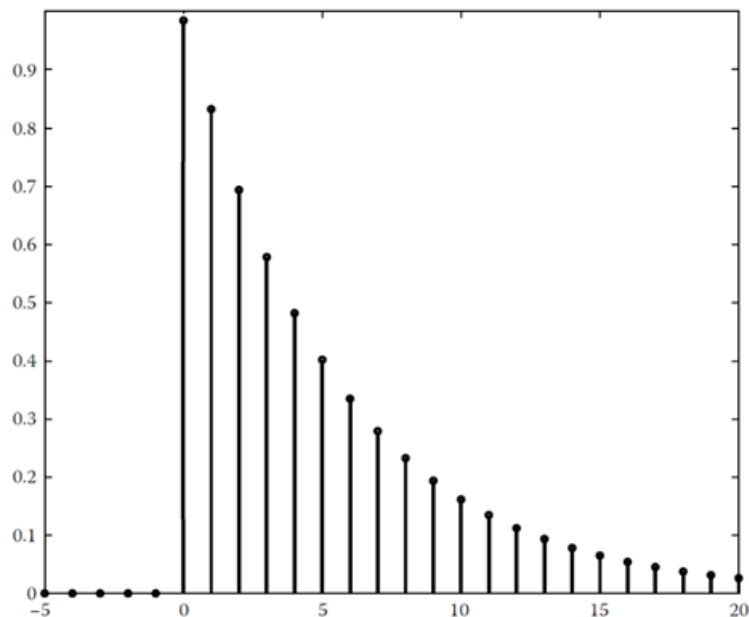
$$F(z) = \frac{4Z}{Z - 2}$$

**Example:**

Calculate the Z-transform and the region of conversion for the following two functions:

$$x[k] = \left(\frac{5}{6}\right)^k u[k] \quad \text{and} \quad y[k] = \left(\frac{5}{6}\right)^{k+5} u[k + 5]$$

For the first function  $x[k]$  takes zero value at  $k < 0$  as shown; So, it called causal function (right defined).



Based on z-transform table:

$$X(z) = \frac{z}{z - \frac{5}{6}} \quad |z| > \frac{5}{6}$$

For the second function  $y[k]$  arises by shifting  $x[k]$  by 5 units to the left.

$$y[k] = x[k + 5]$$

$$y[k] = Z^5 X(z) - Z^5 X(0) - Z^4 X(1) - Z^3 X(2) - Z^2 X(3) - Z X(4)$$

There are some negative values for  $n$  as shown below; So, it called noncausal function.



Considering all initial values  $X[0]=1$ ,  $X[1]=5/6$ ,  $X[2]=25/36$ ,  $X[3]=125/216$  and  $X[4]=625/1296$

Based on z-transform table:

$$Y(z) = z^5 \frac{z}{z - \frac{5}{6}} - z^5 - z^4 \frac{5}{6} - z^3 \frac{25}{36} - z^2 \frac{125}{216} - z \frac{625}{1296} \quad |z| > \frac{5}{6}$$

### 6.2.5 Multiplication by $a^k$

If  $x(t)$  has the z transform  $X(Z)$ , then  $x(t)a^t$  has the z transform  $X(Z/a)$

#### Example:

Find the z-transform of the following function

$$y[k] = a^k u[k]$$

Recall that the z-transform of a sampled step is

$$X(Z) = \frac{Z}{Z - 1} \rightarrow X(a^{-1}Z) = \frac{a^{-1}Z}{a^{-1}Z - 1} = \frac{Z}{Z - a}$$

#### Example:

Find the z-transform of the following function

$$x(n) = (0.5)^n \sin(0.25\pi n) u(n)$$

As we know, the z transform of  $\sin(0.25\pi n)$  is

$$\frac{z \sin(0.25\pi)}{z^2 - 2z \cos(0.25\pi) + 1}$$

Replace each Z by Z/0.5 i.e 2Z

$$X(z) = \frac{2z \sin(0.25\pi)}{(2z)^2 - 2(2z) \cos(0.25\pi) + 1}$$

Dividing num and den by  $(2)^2$

$$X(z) = \frac{0.5 z \sin(0.25\pi)}{z^2 - 2(0.5z) \cos(0.25\pi) + (0.5)^2} = \frac{0.3536z}{z^2 - 0.7071z + 0.25}$$

### 6.2.6 Multiplication by Exponential (complex translation theorem)

If  $x(t)$  has the z transform  $X(Z)$ , then  $x(t)e^{-aT}$  has the z transform  $X(Ze^{aT})$



**Example:**

Find the z-transform of the following function

$$y[k] = e^{-k}u[k]$$

Recall that the z-transform of a sampled step is

$$X(Z) = \frac{Z}{Z - 1} \rightarrow X(e^1Z) = \frac{e^1Z}{e^1Z - 1} = \frac{Z}{Z - e^{-1}}$$

Therefore,

$$Y(Z) = X(e^1Z) = \frac{Z}{Z - e^{-1}}$$

**Example:**

Find the z transform of the following function

$$x(n) = e^{-0.1n} \cos(0.25\pi n) u(n)$$

As we know, the z transform of  $\cos(0.25\pi n)$  is

$$\frac{z(z - \cos(0.25\pi))}{z^2 - 2z \cos(0.25\pi) + 1}$$

Replace each Z by  $Ze^{0.1}$

$$X(z) = \frac{ze^{0.1}(ze^{0.1} - \cos(0.25\pi))}{(ze^{0.1})^2 - 2ze^{0.1} \cos(0.25\pi) + 1}$$

Dividing num and den by  $(e^{0.1})^2$

$$X(z) = \frac{z(z - e^{-0.1}\cos(0.25\pi))}{z^2 - 2ze^{-0.1}\cos(0.25\pi) + (e^{-0.1})^2} = \frac{z(z - 0.6398)}{z^2 - 1.2796z + 0.8187}$$

**6.2.7 Convolution**

Suppose we have two functions along with their respective Z transforms:

$$x_1[k] \leftrightarrow X_1(z) \quad \text{and} \quad x_2[k] \leftrightarrow X_2(z)$$

The convolution property means that

$$Z(x_1[k] * x_2[k]) = X_1(z) * X_2(z)$$

**6.2.8 Differentiation in z-Domain (Complex Differentiation)**

Suppose we have a function with its Z transform:

$$x[k] \leftrightarrow X(z)$$

Then



$$k x[k] \leftrightarrow -z \frac{dX(z)}{dz}$$

We used this property to proof the Z-transform of ramp function.

**Example:**

Calculate the z-transform of the following function:

$$y[k] = k \cdot 2^k u[k]$$

$$x[k] = 2^k u[k] \rightarrow X(Z) = \frac{Z}{Z-2} \rightarrow \frac{dX(z)}{dz} = \frac{(Z-2) \times 1 - Z(1)}{(Z-2)^2} = \frac{-2}{(Z-2)^2}$$

$$Y(Z) = -Z \frac{dX(z)}{dz} = \frac{2Z}{(Z-2)^2}$$

**Example:**

Calculate the z-transform of the following function:

$$y[k] = k \cdot \left(\frac{1}{2}\right)^k u[k-2]$$

As we know:

$$x[k] = \left(\frac{1}{2}\right)^{k-2} u[k-2] \leftrightarrow X(z) = Z^{-2} \frac{Z}{Z-\frac{1}{2}} = \frac{Z^{-1}}{Z-\frac{1}{2}}, \quad |Z| > \frac{1}{2}$$

$$g[k] = \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{k-2} u[k-2] = \left(\frac{1}{2}\right)^2 x[k] \leftrightarrow G(z) = \frac{1}{4} \frac{Z^{-1}}{Z-\frac{1}{2}}, \quad |Z| > \frac{1}{2}$$

Also, we know that:

$$y[k] = k \cdot g[k] \leftrightarrow Y(z) = -z \frac{dG(z)}{dz}$$

$$\frac{dG(z)}{dz} = \frac{1}{4} \frac{\left(Z-\frac{1}{2}\right)(-Z^{-2}) - Z^{-1}(1)}{\left(Z-\frac{1}{2}\right)^2}$$

$$Y(z) = \frac{1}{4} \frac{Z^{-1} \left(Z-\frac{1}{2}\right) + 1}{\left(Z-\frac{1}{2}\right)^2} = \frac{1}{4} \frac{2 - \frac{1}{2}Z^{-1}}{\left(Z-\frac{1}{2}\right)^2}$$

**6.2.9 Initial Value Theorem**

$$\lim_{k \rightarrow 0} x[k] = x[0] = \lim_{Z \rightarrow \infty} X(z)$$



### 6.2.10 Final Value Theorem

$$\lim_{k \rightarrow \infty} x[k] = x[\infty] = \lim_{z \rightarrow 1} (z - 1)X(z)$$

#### Example:

Calculate the initial value and the final value of the following function:

$$Y(z) = \frac{2z^2}{(z - 1)(z - 0.5)(z - 0.25)}$$

$$Y(z) = \frac{2}{(z - 1)\left(1 - \frac{0.5}{z}\right)\left(1 - \frac{0.25}{z}\right)}$$

The initial value is given by:

$$y[0] = \lim_{z \rightarrow \infty} Y(z) = \frac{2}{\infty} = 0$$

The final value is given by:

$$y[\infty] = \lim_{z \rightarrow 1} (z - 1)Y(z) = \lim_{z \rightarrow 1} \frac{2z^2}{(z - 0.5)(z - 0.25)} = \frac{2}{0.5 \times 0.75} = \frac{16}{3}$$

#### Example:

Verify the final value theorem using the  $z$ -transform of a decaying exponential sequence and its limit as  $k$  tends to infinity.

The  $z$ -transform pair of an exponential sequence is

$$\{e^{-akT}\} \xleftrightarrow{\mathcal{Z}} \frac{z}{z - e^{-aT}} \quad \text{with } a > 0.$$

The limit as  $k$  tends to infinity in the time domain is

$$f(\infty) = \lim_{k \rightarrow \infty} e^{-akT} = 0$$

The final value theorem gives

$$f(\infty) = \lim_{z \rightarrow 1} \left(\frac{z - 1}{z}\right) \left(\frac{z}{z - e^{-aT}}\right) = 0$$

#### Example:

Obtain the Z-transform of the following transfer function considering  $T=1$

$$X(s) = \frac{1}{S(S + 1)}$$

Using partial fraction

$$X(s) = \frac{1}{S} - \frac{1}{S + 1}$$



Taking inverse Laplace

$$x(t) = 1 - e^{-t}$$

In discrete form, the function will be

$$x(k) = u(k) - e^{-k}$$

Taking the Z-transform

$$X(z) = \frac{Z}{Z-1} - \frac{Z}{Z-e^{-1}} = \frac{0.6321 Z}{(Z-1)(Z-0.3679)}$$

### **Example:**

Obtain the Z-transform of the following transfer function considering T=1

$$X(s) = \frac{1}{(s+2)(s+1)^2}$$

Using partial fraction

$$X(s) = \frac{2}{s+1} - \frac{1}{(s+1)^2} - \frac{2}{s+2}$$

Taking inverse Laplace

$$x(t) = 2e^{-t} - te^{-t} - 2e^{-2t}$$

Taking the Z-transform

$$X(z) = \frac{2Z}{Z-e^{-1}} - \left\{ -Z \frac{d}{dz} \left( \frac{Z}{Z-e^{-1}} \right) \right\} - \frac{2Z}{Z-e^{-2}}$$

we obtain the z transform from Laplace directly

$$\begin{aligned} X(z) &= \frac{2Z}{Z-e^{-1}} - \frac{Ze^{-1}}{(Z-e^{-1})^2} - \frac{2Z}{Z-e^{-2}} \\ X(z) &= \frac{2Z^2 - 3Ze^{-1}}{(Z-e^{-1})^2} - \frac{2Z}{Z-e^{-2}} \\ X(z) &= \frac{2Z^2 - 1.1036Z}{(Z-0.3679)^2} - \frac{2Z}{Z-0.13534} \end{aligned}$$

## **7. Z-Transform Table**

Instead of performing summation to get the Z-transform as explained in

$$X(z) = \sum_{k=-\infty}^{n=\infty} x[k].Z^{-k}$$

We can use the following table:





No.	Signal $x(n), n \geq 0$	z-Transform $Z(x(n)) = X(z)$	Region of Convergence
1	$x(n)$	$\sum_{n=0}^{\infty} x(n) z^{-n}$	
2	$\delta(n)$	1	Entire z-plane
3	$a u(n)$	$\frac{az}{z-1}$	$ z  > 1$
4	$n u(n)$	$\frac{z}{(z-1)^2}$	$ z  > 1$
5	$n^2 u(n)$	$\frac{z(z+1)}{(z-1)^3}$	$ z  > 1$
6	$a^n u(n)$	$\frac{z}{z-a}$	$ z  >  a $
7	$e^{-na} u(n)$	$\frac{z - e^{-a}}{az}$	$ z  > e^{-a}$
8	$n a^n u(n)$	$\frac{z}{(z-a)^2}$	$ z  >  a $
9	$\sin(an) u(n)$	$\frac{z \sin(a)}{z^2 - 2z \cos(a) + 1}$	$ z  >  1 $
10	$\cos(an) u(n)$	$\frac{z(z - \cos(a))}{z^2 - 2z \cos(a) + 1}$	$ z  >  1 $
11	$a^n \sin(bn) u(n)$	$\frac{[a \sin(b)] z}{z^2 - [2a \cos(b)]z + a^2}$	$ z  >  a $
12	$a^n \cos(bn) u(n)$	$\frac{z [z - a \cos(b)]}{z^2 - [2a \cos(b)]z + a^2}$	$ z  >  a $
13	$e^{-an} \sin(bn) u(n)$	$\frac{[e^{-a} \sin(b)] z}{z^2 - [2e^{-a} \cos(b)]z + e^{-2a}}$	$ z  > e^{-a}$
14	$e^{-an} \cos(bn) u(n)$	$\frac{z [z - e^{-a} \cos(b)]}{z^2 - [2e^{-a} \cos(b)]z + e^{-2a}}$	$ z  > e^{-a}$

## 8. Inverse Z-Transform

The implementation of z-transform results in the transformation from the discrete-time domain to z-domain. The opposite procedure is implemented with the aid of the inverse z-transform. The inverse z-transform is defined by

$$x[n] = Z^{-1}\{X(z)\} = \frac{1}{2\pi j} \oint_c X(z) z^{n-1} dz$$

Where c is the closed contour within the region of convergence of X(z).

There are 3 methods for calculating the inverse transform of a function X(z)



1. Method of *power series expansion (Long Division)*
2. Method of *partial fraction expansion*
3. Method of *complex integration* (via the residue theorem)

**Example:**

Find the first 6 coefficients of the function  $x[n]$  when

$$X(z) = \frac{4Z^{-1}}{Z^{-2} - 2Z^{-1} + 2}$$

Convert all powers of  $Z$  to +ve powers

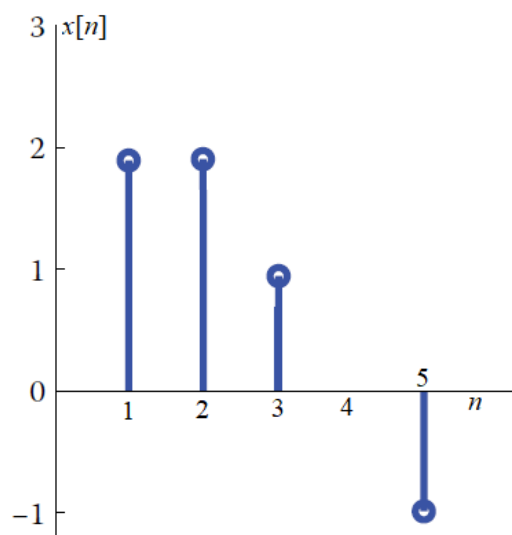
$$X(z) = \frac{4Z}{1 - 2Z + 2Z^2} = \frac{4Z}{2Z^2 - 2Z + 1}$$

By long division:

$$\begin{array}{r}
 2z^{-1} + 2z^{-2} + z^{-3} - z^{-5} + \dots \\
 \underline{2z^2 - 2z + 1} \quad 4z \\
 -4z + 4 - 2z^{-1} \\
 \hline
 4 - 2z^{-1} \\
 -4 + 4z^{-1} - 2z^{-2} \\
 \hline
 2z^{-1} + 2z^{-2} \\
 -2z^{-1} - 2z^{-2} - 2z^{-3} \\
 \hline
 -2z^{-3}
 \end{array}$$

$$X(z) = 2z^{-1} + 2z^{-2} + z^{-3} - z^{-5} + \dots$$

$$x[n] = \{0, 2, 2, 1, 0, -1, \dots\}$$





**Example:**

Find the first 5 coefficients of the function  $x[n]$  when

$$X(z) = \frac{z^2 - 1}{z^3 + 2z + 4}$$

By long division:

$$\begin{array}{r}
 z^3 + 2z + 4 \overline{) z^2 - 1} \\
 \underline{z^2 + 2 + 4z^{-1}} \\
 -3 - 4z^{-1} \\
 \underline{-3 - 6z^{-2} - 12z^{-3}} \\
 -4z^{-1} + 6z^{-2} + 12z^{-3} \\
 \underline{-4z^{-1} - 8z^{-3} - 16z^{-4}} \\
 6z^{-2} + 20z^{-3} + 16z^{-4}
 \end{array}$$

$$\begin{aligned}
 X(z) &= z^{-1} - 3z^{-3} - 4z^{-4} + \dots \\
 x[n] &= \{0, 1, 0, -3, -4, \dots\}
 \end{aligned}$$

**Example:**

Consider the sequence

$$f(k) = \{1, 3, 2, 0, 4, 0, 0, 0\}$$

Find  $F(z)$ .

$$\begin{aligned}
 F(z) &= \sum_{k=0}^{\infty} f(k) z^{-k} \\
 F(z) &= [1 \times z^0 + 3 \times z^{-1} + 2 \times z^{-2} + 4 \times z^{-4}] = 1 + 3z^{-1} + 2z^{-2} + 4z^{-4}
 \end{aligned}$$

**Example:**

Compute the inverse  $z$ -transform of the function

$$F(z) = \frac{z}{(z + 0.1)(z + 0.2)(z + 0.3)}$$

Dividing both sides by  $(z)$ :

$$\frac{F(z)}{z} = \frac{1}{(z + 0.1)(z + 0.2)(z + 0.3)}$$

Using partial fraction:



$$\frac{F(z)}{Z} = \frac{A}{Z + 0.1} + \frac{B}{Z + 0.2} + \frac{C}{Z + 0.3}$$

A = 50, B = -100 and C = 50

$$F(Z) = \frac{50Z}{Z + 0.1} - \frac{100Z}{Z + 0.2} + \frac{50Z}{Z + 0.3}$$

Using Inverse Z-Transform

$$f[k] = 50(-0.1)^n u(n) - 100(-0.2)^n u(n) + 50(-0.3)^n u(n), \quad k > 0$$

**Example:**

Compute the inverse z-transform of the function

$$X(z) = \frac{Z^{-2} + Z^{-1} + 1}{0.2Z^{-2} + 0.9Z^{-1} + 1}$$

Convert all powers of Z to +ve powers

$$X(z) = \frac{Z^2 + Z^1 + 1}{Z^2 + 0.9Z + 0.2} = \frac{Z^2 + Z^1 + 1}{(Z + 0.4)(Z + 0.5)}$$

Divide both sides by Z

$$\frac{X(z)}{Z} = \frac{Z^2 + Z^1 + 1}{Z(Z + 0.4)(Z + 0.5)}$$

Using partial fraction:

$$\frac{X(z)}{Z} = \frac{Z^2 + Z^1 + 1}{Z(Z + 0.4)(Z + 0.5)} = \frac{A}{Z} + \frac{B}{Z + 0.4} + \frac{C}{Z + 0.5}$$

$$Z^2 + Z^1 + 1 = A(Z + 0.4)(Z + 0.5) + BZ(Z + 0.5) + CZ(Z + 0.4)$$

At Z=0      1 = 0.2 A              → A=5

At Z=-0.4    0.76 = -0.04 B           → B= -19

At Z=-0.5    0.75 = 0.05 C                → C= 15

$$\frac{X(z)}{Z} = \frac{5}{Z} - \frac{19}{Z + 0.4} + \frac{15}{Z + 0.5}$$

$$X(z) = 5 - \frac{19Z}{Z + 0.4} + \frac{15Z}{Z + 0.5}$$

Taking inverse Z transform:



$$x[n] = 5\delta(n) - 19(-0.4)^n u(n) + 15(-0.5)^n u(n)$$

**Example:**

Compute the inverse  $z$ -transform of the function

$$X(z) = \frac{Z^3 + 2Z + 1}{(Z - 0.1)(Z^2 + Z + 0.5)}$$

As the quadratic term (in denominator) gives complex roots, no need to factorize it.

Divide both sides by  $Z$ :

$$\frac{X(z)}{Z} = \frac{Z^3 + 2Z + 1}{Z(Z - 0.1)(Z^2 + Z + 0.5)}$$

Using partial fractions:

$$\frac{X(z)}{Z} = \frac{Z^3 + 2Z + 1}{Z(Z - 0.1)(Z^2 + Z + 0.5)} = \frac{A}{Z} + \frac{B}{Z - 0.1} + \frac{CZ + D}{Z^2 + Z + 0.5}$$

$$A = -20$$

$$B = 19.689$$

$$C = 1.311$$

$$D = -1.557$$

$$X(Z) = -\frac{20Z}{Z} + \frac{19.689Z}{Z - 0.1} + \frac{1.311Z^2 - 1.557Z}{Z^2 + Z + 0.5}$$

$$a^n \cos(bn) u(n) \rightarrow \frac{z [z - a \cos(b)]}{z^2 - [2a \cos(b)]z + a^2}$$

$$a^n \sin(bn) u(n) \rightarrow \frac{[a \sin(b)] z}{z^2 - [2a \cos(b)]z + a^2}$$

$$\frac{1.311Z(Z - 1.1876)}{Z^2 + Z + 0.5} = \frac{1.311Z(Z - 0.7071 \times (-0.7071) - 1.6876)}{Z^2 - 2 \times 0.7071 \times (-0.7071)Z + (0.7071)^2}$$

$$\frac{1.311Z(Z - 0.7071 \times (-0.7071))}{Z^2 - 2 \times 0.7071 \times (-0.7071)Z + (0.7071)^2} = \frac{2.2124Z}{Z^2 - 2 \times 0.7071 \times (-0.7071)Z + (0.7071)^2}$$

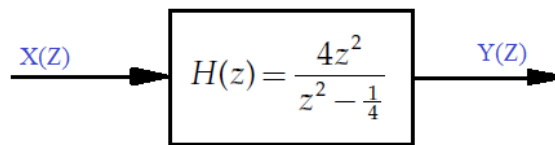
$$\frac{2.2124Z}{Z^2 - 2 \times 0.7071 \times (-0.7071)Z + (0.7071)^2} = \frac{4.4248 \times 0.7071 \times 0.7071Z}{Z^2 - 2 \times 0.7071 \times (-0.7071)Z + (0.7071)^2}$$



$$x[n] = -20\delta(n) + 19.689(0.1)^n u(n) + 1.311(0.7071)^n \cos\left(\frac{3\pi}{4}n\right) u(n) \\ - 4.4248(0.7071)^n \sin\left(\frac{3\pi}{4}n\right) u(n)$$

**Example:**

For the discrete-time system shown below, if  $x[n]=n u(n)$ , calculate the output  $y(n)$ .



$$X(z) = \frac{Z}{(Z - 1)^2}$$

$$Y(z) = X(Z)H(Z) = \frac{Z}{(Z - 1)^2} \frac{4Z^2}{Z^2 - 0.25}$$

$$\frac{Y(z)}{Z} = \frac{1}{(Z - 1)^2} \frac{4Z^2}{Z^2 - 0.25} = \frac{4Z^2}{(Z - 0.5)(Z + 0.5)(Z - 1)^2}$$

$$\frac{Y(z)}{Z} = \frac{4Z^2}{(Z - 0.5)(Z + 0.5)(Z - 1)^2} = \frac{A}{Z - 0.5} + \frac{B}{Z + 0.5} + \frac{C}{Z - 1} + \frac{D}{(Z - 1)^2}$$

A= 4, B= -4/9, C= -32/9 and D = 16/3

$$Y(z) = \frac{4Z}{Z - 0.5} - \frac{\frac{4}{9}Z}{Z + 0.5} - \frac{\frac{32}{9}Z}{Z - 1} + \frac{\frac{16}{3}Z}{(Z - 1)^2}$$

$$y[n] = 4(0.5)^n u(n) - \frac{4}{9}(-0.5)^n u(n) - \frac{32}{9}u(n) + \frac{16}{3}n u(n)$$

**8. Solving Difference Equation**

**Example:**

Solve the following linear difference equation,

$$x(k + 2) - \frac{3}{2}x(k + 1) + \frac{1}{2}x(k) = 1(k)$$

With initial conditions  $x(0) = 1, x(1) = \frac{5}{2}$

1. Taking z-transform of given equation

$$[z^2 X(z) - z^2 x(0) - z x(1)] - \frac{3}{2}[z X(z) - z x(0)] + \frac{1}{2}X(z) = \frac{z}{z - 1}$$



2. Solve for  $X(z)$

$$\left[z^2 - \frac{3}{2}z + \frac{1}{2}\right]X(z) = \frac{z}{z-1} + z^2 + \left(\frac{5}{2} - \frac{3}{2}\right)z$$

$$X(z) = \frac{z[1 + (z+1)(z-1)]}{(z-1)(z-1)(z-0.5)} = \frac{z^3}{(z-1)^2(z-0.5)}$$

3. Inverse z-transform

$$\frac{X(z)}{z} = \frac{z^2}{(z-1)^2(z-0.5)} = \frac{A}{(z-1)^2} + \frac{B}{z-1} + \frac{C}{z-0.5}$$

$$X(z) = \frac{2z}{(z-1)^2} + \frac{z}{z-0.5}$$

$$x(k) = 2k + (0.5)^k$$

**Example:**

A relaxed (zero initial conditions) DSP system is described by the difference equation

$$y(n] + 0.1y[n - 1) - 0.2y[n - 2) = x[n) + x[n - 1)$$

Determine the impulse response  $y(n)$  due to the impulse sequence  $x(n) = \delta(n)$ .

Taking z-transform of both sides of the given equation, we get

$$Z(y(n)) + 0.1Z(y(n - 1)) - 0.2Z(y(n - 2)) = Z(x(n)) + Z(x(n - 1))$$

We have

$$Z(y(n)) = Y(z)$$

$$Z(x(n)) = X(z)$$

Using shift theorem, we have

$$Z(x(n - 1)) = z^{-1}X(z)$$

Also we can apply shift theorem for  $y$  in case of zero initial conditions, i.e.,

$$Z(y(n - 1)) = z^{-1}Y(z)$$

$$Z(y(n - 2)) = z^{-2}Y(z)$$

Putting these values in Equation (1), we have

$$Y(z) + 0.1z^{-1}Y(z) - 0.2z^{-2}Y(z) = X(z) + z^{-1}X(z)$$



$$\Rightarrow Y(z)(1 + 0.1z^{-1} - 0.2z^{-2}) = X(z)(1 + z^{-1})$$

As  $x(n) = \delta(n)$  therefore, (from Table 1),  $X(z) = 1$ . The above equation can now be written as

$$Y(z) = \frac{(1 + z^{-1})}{(1 + 0.1z^{-1} - 0.2z^{-2})}$$

Multiplying both the numerator and the denominator with  $z^2$  we get

$$Y(z) = \frac{z(z + 1)}{(z^2 + 0.1z - 0.2)}$$

The denominator can be factorized as

$$\begin{aligned} Y(z) &= \frac{z(z + 1)}{(z^2 + 0.5z - 0.4z - 0.2)} = \frac{z(z + 1)}{(z(z + 0.5) - 0.4(z + 0.5))} \\ &= \frac{z(z + 1)}{(z + 0.5)(z - 0.4)} \end{aligned}$$

$$\Rightarrow \frac{Y(z)}{z} = \frac{(z + 1)}{(z + 0.5)(z - 0.4)}$$

The right hand side of the above equation is a proper rational polynomial, with the denominator polynomial having distinct poles, therefore, it can be written into partial fractions as

$$\frac{Y(z)}{z} = \frac{A}{(z + 0.5)} + \frac{B}{(z - 0.4)}$$

To find out the unknown constants **A** and **B**, we use partial fractions

$$A = -0.5556$$

$$B = 1.5556$$

$$\frac{Y(z)}{z} = \frac{-0.5556}{(z + 0.5)} + \frac{1.5556}{(z - 0.4)}$$

$$Y(z) = \frac{-0.5556 z}{(z + 0.5)} + \frac{1.5556 z}{(z - 0.4)}$$

Taking inverse z-transform of both sides





$$\begin{aligned} y(n) &= Z^{-1}(Y(z)) = Z^{-1}\left(\frac{-0.5556z}{(z+0.5)}\right) + Z^{-1}\left(\frac{1.5556z}{(z-0.4)}\right) \\ &= (-0.5556)Z^{-1}\left(\frac{z}{(z-(-0.5))}\right) + (1.5556)Z^{-1}\left(\frac{z}{(z-0.4)}\right) \\ &= (-0.5556)(-0.5)^n u(n) + (1.5556)(0.4)^n u(n) \end{aligned}$$

Thus the output signal is

$$y(n) = (-0.5556)(-0.5)^n u(n) + (1.5556)(0.4)^n u(n)$$

**Example:**

Solve for  $x(k)$  of the difference equation:

$$x(k+2) - 1.3679x(k+1) + 0.3679x(k) = 0.3679u(k+1) + 0.2642u(k)$$

Known that:  $u(k) = 0$  for  $k \leq 0$ ,  $u(0) = 1$ ,  $u(1) = 0.2142$ ,  $u(2) = -0.2142$ ,  $u(k) = 0$   $k=3,4,\dots$

Also  $x(k) = 0$  for  $k \leq 0$

From the given values of the input function, we can get an expression for  $U(z)$

$$U(z) = 1 + 0.2142z^{-1} - 0.2142z^{-2}$$

Taking the Z transform for both sides of the difference equation

$$\begin{aligned} [z^2 X(z) - z^2 x(0) - z x(1)] - 1.3679[z X(z) - z x(0)] + 0.3679X(z) \\ = 0.3679[z U(z) - z u(0)] + 0.2642U(z) \end{aligned}$$

In the above equation  $x(0) = 0$ ,  $u(0) = 1$

$$\begin{aligned} z^2 X(z) - z x(1) - 1.3679z X(z) + 0.3679X(z) \\ = 0.3679z U(z) - 0.3679z + 0.2642U(z) \end{aligned}$$

It is clear that  $x(1)$  is unknown. To calculate its value, we substitute  $k = -1$  in the given difference equation

$$\begin{aligned} x(1) - 1.3679x(0) + 0.3679x(-1) &= 0.3679u(0) + 0.2642u(-1) \\ x(1) &= 0.3679 \end{aligned}$$

$$X(z)(z^2 - 1.3679z + 0.3679) - 0.3679z = U(z)(0.3679z + 0.2642) - 0.3679z$$

$$X(z)(z^2 - 1.3679z + 0.3679) = U(z)(0.3679z + 0.2642)$$

$$X(z) = \frac{0.3679z + 0.2642}{z^2 - 1.3679z + 0.3679} U(z)$$



$$X(z) = \frac{0.3679z + 0.2642(1 + 0.2142z^{-1} - 0.2142z^{-2})}{z^2 - 1.3679z + 0.3679}$$

$$X(z) = \frac{0.3679z + 0.343 - 0.0222z^{-1} - 0.0566z^{-2}}{z^2 - 1.3679z + 0.3679}$$

$$X(z) = \frac{0.3679z^3 + 0.343z^2 - 0.0222z - 0.0566}{z^4 - 1.3679z^3 + 0.3679z^2}$$

By long division we obtain:

$$X(z) = 0.3679z^{-1} + 0.8463z^{-2} + z^{-3} + z^{-4} + \dots$$

$$x(0) = 0$$

$$x(1) = 0.3679$$

$$x(2) = 0.8463$$

$$x(3) = 1.0$$

$$x(4) = 1.0$$

## 9. Matlab Code for Z-Transform and Inverse Z-Transform

ztrans → z-transform

iztrans → inverse z-transform

Compute the z-transform of  $f(k) = 2^k$  using MATLAB

```
syms k, z
```

```
f = 2^k;
```

```
ztrans(f)
```

```
simplify(ans)
```

$$F(z) = z/(z - 2)$$

To verify the above result, we calculate the inverse z-transform of  $F(z) = z/(z - 2)$ ;

```
syms k, z
```

```
F = z/(z - 2);
```

```
iztrans(F)
```

```
ans = 2^k
```

### References:



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